Extra Practice Problems 2

Here are some more practice problems to work through. We'll release solutions on Monday.

First-Order Logic

For each first-order statement below, write a short English sentence that describes what that sentence says. Then, determine whether the statement is true or false. No proofs are necessary.

(You might want to read over the checkpoint problem for Problem Set Four before starting this problem, since it introduces some terminology that might make it easier to translate these statements.)

- $\exists S.\ (Set(S) \land \forall x.\ x \notin S)$
- $\forall x.\ \exists S.\ (Set(S) \land x \notin S)$
- $\forall S.\ (Set(S) \to \exists x.\ x \notin S)$
- $\forall S.\ (Set(S) \land \exists x.\ x \notin S)$
- $\exists S.\ (Set(S) \land \exists x.\ x \notin S)$
- $\exists S.\ (Set(S) \to \forall x.\ x \in S)$
- $\exists S.\ (Set(S) \land \forall x.\ x \notin S \land \forall \text{T}.\ (Set(T) \land S \neq T \to \exists x.\ x \in T))$
- $\exists S.\ (Set(S) \land \forall x.\ x \notin S \land \exists \text{T}.\ (Set(T) \land \forall x.\ x \notin T \land S \neq T))$
- $\exists S.\ (Set(S) \land \forall x.\ x \notin S) \land \exists \text{T}.\ (Set(T) \land \forall x.\ x \notin T))$

Graph Theory

As you've seen on the problem sets, a tournament graph is a directed graph with $n \geq 1$ nodes where each pair of two different nodes has exactly one edge between them. (Intuitively, the nodes are players, and the edges represent the outcomes of each game).

As you saw in lecture, every tournament has at least one victory chain, a way of lining up the players from left to right so that everyone beat the player immediately after them and lost to the player immediately before them. However, a victory chain doesn't guarantee anything else about the ordering of the players. For example, in a victory chain p_1, p_2, p_3, it's possible that p_3 beat p_1.

A *strong victory chain* is a way of ordering the players in a tournament so that each player appears only once in the ordering, each player lost to *every* player that came before them in the chain, and each player beat *every* player that comes after them in the chain.

Prove that if a tournament graph has a strong victory chain, then it doesn't contain any cycles.

Induction

A doctor has prescribed a patient medicine that is absorbed into the bloodstream. The medicine has a *half-life* of one hour, meaning that each hour, half of the medicine in the patient's bloodstream will be removed by her body. For example, if the patient had 5mg of the medicine in her bloodstream at 6:00PM, then at 7:00PM she would have 2.5mg of the medicine in her bloodstream.

Suppose that the doctor gives the patient 1mg of the medicine at the start of every hour, all of which is immediately absorbed into her bloodstream. You are concerned because each time the patient receives a dose, some amount of the medicine will still be left in her bloodstream. Wouldn't this give the patient a dangerous amount of medicine?

Fortunately, now that you've taken a course in discrete math, you can determine exactly how much medicine will be in the patient's bloodstream, which will help you determine whether she will ever have a dangerous amount of the medicine in her blood.

Let c_n denote the amount of active medication in the patient's body n hours after the first dose has been administered. The first dose is 1mg, so c_0 = 1mg. One hour later, half of the medicine will have been cleared from her bloodstream, leaving 0.5mg, and the patient will receive 1mg more medicine, bringing the total up to 1.5mg. Thus c_1 = 1.5mg. An hour after that, half that medicine will have been cleared from her bloodstream, leaving 0.75mg of medicine in her bloodstream, and the patient will then receive another 1mg of medicine, bringing the total up to 1.75mg. Thus c_2 = 1.75mg.

i. Write a recurrence relation for c_n.

ii. Prove, by induction, that $c_n = (2 - 1/2^n)$mg for all $n \in \mathbb{N}$. This proves that the patient will never have more than 2mg of medicine in her bloodstream, even if she continues to take 1mg doses every hour.